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# Some remarks on Hele-Shaw flow and viscous tails

By J. BUCKMASTER

New York University, University Heights, N.Y.

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A simple model is suggested to describe flow in a Hele-Shaw cell when the Hele-Shaw parameter $\Lambda$ is not necessarily small. The averaged flow is potential with a conservative body force proportional to the local velocity. The elementary ramifications of this are deduced and comparisons made with experiment. In particular no separation is predicted if $\Lambda$ is less than $O(1)$, in agreement with experiment. Furthermore, the separation cavities occurring for large $\Lambda$ are completely stagnant. The theory predicts attached viscous shear layers in the wake of a lifting body, reminiscent of certain MHD problems. These tails were not observed experimentally.

## 1. Introduction

It has been known for over 70 years that slow viscous flow in a narrow gap can be described by a velocity potential. Hele-Shaw (1897, 1898*a*, *b*) noticed this experimentally and Stokes (1898) established it theoretically. The Navier–Stokes equations are

$$\left.\begin{aligned}(\mathbf{q}\,.\nabla)\,\mathbf{q} &= -\nabla p+\frac{1}{R}\left(\frac{\partial^2\mathbf{q}}{\partial x^2}+\frac{\partial^2\mathbf{q}}{\partial y^2}+\frac{\partial^2\mathbf{q}}{\partial z^2}\right),\\ \nabla\,.\mathbf{q} &= 0.\end{aligned}\right\} \tag{1}$$

If the bounding walls of the Hele-Shaw cell are planes of constant $z$ then, provided we are not too close to any body confined within the walls, the dominant viscous term is

$$\frac{1}{R}\frac{\partial^2\mathbf{q}}{\partial z^2}.$$

If the non-dimensional cell thickness is $O(h)$ this term is $O(\Lambda^{-1} = (Rh^2)^{-1})$ and completely dominates the inertia terms when $\Lambda$ is very small. The characteristic length for our problem is here taken to be a typical $x$-$y$ dimension of any body inserted in the cell. Hele-Shaw flow is then described by the equations

$$\left.\begin{aligned}0 &= -\nabla p+\frac{1}{R}\frac{\partial^2\mathbf{q}}{\partial z^2},\\ \nabla\,.\mathbf{q} &= 0,\end{aligned}\right\} \tag{2}$$

with solution
$$\mathbf{q} = -\tfrac{1}{2}\Lambda(1-z^2/h^2)\,\nabla p, \tag{3}$$

where $p$ satisfies the two-dimensional Laplace equation. This solution, first obtained by Stokes (1898), does not satisfy the no-slip condition on any two-dimensional body within the cell. The other viscous terms have to be reinstated to do this and lead to a boundary layer of thickness $O(h)$. This boundary layer is also inertialess so that it does not separate. The total effect is that the flow exterior to the boundary layer is classical potential flow unmarred by wakes or unsteadiness.

If $\Lambda$ is not small, in particular if $\Lambda \rightarrow \infty$, and $R$ is not small then the usual two-dimensional flow picture is appropriate. The flow over bluff bodies does not then resemble the classical solution everywhere.

It is of interest to describe the transition between these two extremes. It can be expected that as $\Lambda$ is increased from very small values, that is as the inertia terms play a bigger role, then the nature of the flow changes and in particular, at some critical value of $\Lambda$, separation will first occur at the rear stagnation point. Riegels (1938) was very much concerned with the flow for moderate values of $\Lambda$. His photographs of flow over a circular cylinder show that separation first occurs when $\Lambda = O(1)$. At the same time the dye streaks, especially in the rear half of the flow, thicken and indicate the growing importance of three-dimensional effects other than the parabolic shape factor of (3). Riegels did a perturbation analysis for small $\Lambda$ which he hoped to use to indicate what would happen for moderate $\Lambda$ with $h$ still very small. In other words, he was concerned with the limits

$$h \rightarrow 0, \quad R \rightarrow \infty, \quad \Lambda = O(1). \tag{4}$$

Not surprisingly, the comparison between theory and experiment was not very successful since $\Lambda$ in the experiments was much too large to be described by a small $\Lambda$ solution.

Thompson (1968) was attracted by a non-uniformity that Riegels experienced in his small $\Lambda$ expansions close to the body. He resolved this in a satisfactory way when $\Lambda = O(h^2)$, i.e. $R = O(1)$. The leading perturbation to the main body of the flow is then the $O(h)$ displacement thickness effect with the inertia terms first making a contribution at the $O(h^2)$ term. This is no longer Riegels problem however, as equation (4) reminds us. Thompson's analysis can be carried out with little modification when $\Lambda = O(h)$ so that at least the inertia terms play a leading role in the perturbation but the limit (4) is a more complicated problem that remains to be solved.

However, there is an alternative method of achieving some understanding of what happens when $\Lambda = O(1)$. This is in the nature of an irrational approximation. A distinctive feature of Riegels photographs is that $\Lambda$ has to be somewhat larger than 1 before the smearing of the dye-lines becomes very serious. This suggests that for small to moderate values of $\Lambda$ the flow can still be represented well enough by a function of $x$ and $y$ together with a shape factor depending on $z$. The purpose of this note is to explore the elementary ramifications of this observation and to make some comparisons with a problem in magnetohydrodynamics.

## 2. Averaged equations

If the $x$ component of the momentum equation (1) is multiplied by $dz$ and integrated between the cell walls, it can be written as

$$\int_{-h}^{+h} dz \left[\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv)\right] = -\frac{\partial}{\partial x}\int_{-h}^{+h} p\,dz + \frac{1}{R}\int_{-h}^{+h} \frac{\partial^2 u}{dz^2}\,dz, \tag{5}$$

where the $z$ component of velocity $w$ has been eliminated by using the continuity equation. Encouraged by Riegels experiments we approximate the flow field by

$$p = \bar{p}(x,y), \quad \left.\begin{aligned} u &= \bar{u}(x,y)\,(1-z^2/h^2),\\ v &= \bar{v}(x,y)\,(1-z^2/h^2).\end{aligned}\right\} \tag{6}$$

Substitution of (6) into (5) then yields an equation for the averaged flow variables. This equation governs the best (in some sense) quadratic approximation to the flow field. It is

$$\frac{8}{15}\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x}-\frac{2\bar{u}}{\Lambda}+\frac{2}{3R}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\bar{u} \tag{7}$$

with a similar equation for $\bar{v}$. The mean vorticity satisfies

$$\frac{8}{15}\left(\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}\right)\left(\frac{\partial \bar{u}}{\partial y}-\frac{\partial \bar{v}}{\partial x}\right) = -\frac{2}{\Lambda}\left(\frac{\partial \bar{u}}{\partial y}-\frac{\partial \bar{v}}{\partial x}\right)+\frac{2}{3R}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial \bar{u}}{\partial y}-\frac{\partial \bar{v}}{\partial x}\right). \tag{8}$$

If the Reynolds number is large, then outside the boundary layer on the body

$$\frac{\partial \bar{u}}{\partial y}-\frac{\partial \bar{v}}{\partial x} = 0 \tag{9}$$

so that the mean flow is potential. The mean flow satisfies the two-dimensional continuity equation, of course. This result is satisfying since it is exactly correct in the two limits $\Lambda \to 0$ and $\Lambda \to \infty$. In addition, comparison of Riegels' experimental streamlines (for $\Lambda = 1$) with potential flow shows quite reasonable agreement. Over the forward half of the flow-field the difference is negligible. Riegels' photograph reveals a slight asymmetry about the $y$ axis, the streamlines pinching in a little faster to the rear of the body. This introduces a slight discrepancy with the theory over the rear half of the flow field. The smallness of this discrepancy is apparent from the difficulty in detecting the experimental asymmetry without measurement.

The mean flow field is not just another example of potential flow. The body force $(15/4\Lambda)\,(\bar{u},\bar{v})$ is conservative but it plays an important role in the modified Bernoulli equation, which is

$$\tfrac{1}{2}q_s^2+\tfrac{15}{8}\bar{p}+\frac{15}{4\Lambda}\phi = \text{constant along a streamline.} \tag{10}$$

$\phi$ is the velocity potential defined by

$$\mathbf{q} = \nabla\phi$$

and is not necessarily single-valued, of course.

The ramifications of this are amusing but before discussing them it is worth pointing out the connexions between equation (7) and other flow situations. A particular limiting form of filter bed leads to an identical system of equations in fact. Suppose the filter bed is composed of particles so small that the drag they impart to the fluid is given by Stokes law. Then the body force per unit volume is proportional to

$$\mu naq,$$

where $n$ is the particle number density, $a$ is a typical particle dimension, $\mu$ is the fluid viscosity and $q$ its velocity. Consider now the limits

$$n \to \infty, \quad a \to 0, \quad na \to \text{const.}, \quad na^3 \to 0.$$

The volume occupied by the filter particles is then zero, but they impart a body-force to the fluid precisely as in (7) and its companion.

The other comparison we would like to draw is not exact, but the equations, whilst different, lead to similar phenomena. We have in mind the magnetohydrodynamic equations in the limit of zero magnetic Reynolds number but finite interaction parameter. If the magnetic field is uniform and in the $y$ direction the body force (Lorentz force) is $(-Nu, 0)$ where $N$ is a constant. This force is rotational and so the flow is not potential but there are similar effects as we will point out in the appropriate places.

## 3. Separation

The boundary-layer equations associated with (7) are

$$\left.\begin{aligned} \bar{u}_s \frac{\partial \bar{u}_s}{\partial s} + \bar{u}_n \frac{\partial \bar{u}_s}{\partial n} &= \bar{U}\frac{d\bar{U}}{ds} + \frac{15}{4\Lambda}(\bar{U} - \bar{u}_s) + \frac{5}{4R}\frac{\partial^2 \bar{u}_s}{\partial n^2}, \\ \frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} &= 0. \end{aligned}\right\} \qquad (11)$$

Equations (11) are identical to the boundary-layer equations in the MHD problem provided the magnetic field is everywhere perpendicular to the layer. The body force $(\bar{U} - \bar{u}_s)$ acts as a favourable pressure gradient and can completely suppress separation provided $\Lambda$ is small enough. At separation $\partial \bar{u}_s / \partial n = 0$ but $u_s$ is strictly positive in some neighbourhood of the wall so that at separation

$$\frac{\partial^2 \bar{u}_s}{\partial n^2} \geqslant 0.$$

Equation (11) then implies that

$$\frac{d\bar{U}}{ds} + \frac{15}{4\Lambda} \leqslant 0 \quad \text{at separation.} \qquad (12)$$

Clearly, for a given $\bar{U}$ (i.e. a given exterior potential flow) $\Lambda$ can always be chosen small enough that the inequality (12) is never satisfied and so separation cannot take place. To deal with a specific case, suppose we have a circular cylinder for which

$$\bar{U} = 2 \sin s \quad (0 \leqslant s \leqslant \pi).$$

Equation (12) implies then that at separation

$$\cos s \leqslant \frac{-15}{8\Lambda}$$

and separation will not occur if

$$\Lambda < \tfrac{15}{8}. \qquad (13)$$

Riegels experiments imply that $\Lambda$ must, in fact, be closer to 1 to completely inhibit separation but the discrepancy is not too severe. It might be due to the fact that whilst the $z$ averaging is reasonable in the main body of the flow it is suspect whenever the velocity gradient is of order $O(h)$ or greater. On the other hand, the separation criterion can be established by an alternative argument that apparently does not make use of the averaged boundary-layer equations. To do this we make direct use of the modified Bernoulli relation, (10). The presence of $\phi$ implies that there can be no circulation around a closed inviscid streamline. (An inviscid streamline is one that does not plunge at some point into a viscous boundary layer.) This result is also true for the MHD problem since the Lorentz force $(-Nu,\ 0)$ always opposes the motion and therefore

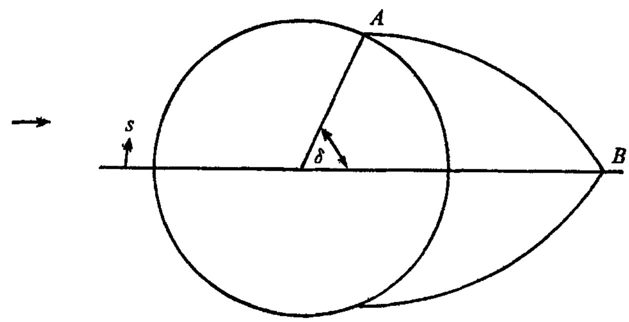


FIGURE 1. Separation bubble behind a circular cylinder.

circulation in one direction would imply a pressure discontinuity. This implies that the wake-cavities that are formed when $\Lambda$ is large enough to permit separation, are true dead-water regions in which the pressure does not change across the streamline bounding the cavity. If this is true (severe three-dimensional effects could invalidate it) then on an inviscid streamline adjacent to the cavity, equation (10) implies

$$\frac{d\bar{q}_s}{ds}+\frac{15}{4\Lambda}=0$$

and the velocity falls steadily along such a streamline. Consider figure 1, where a wake cavity is sketched at the back of a circular cylinder. Suppose that the speed at $A$ is $2\sin s_A$. From $A$ to $B$ the speed falls at the rate $(-\frac{15}{4}\Lambda)$ and the least value it can have at $B$ is zero. It follows that the maximum length of $AB$ is $(8\Lambda/15)\sin s_A$ and this must satisfy the inequality

$$\frac{8\Lambda}{15}\sin\delta \geqslant \delta \tag{14}$$

since the streamline lies outside the body. This inequality is only possible if $\Lambda \geqslant \frac{15}{8}$ in agreement with (13). The speed at $A$ is not $2\sin s_A$ of course, but it is in the limit $s_A \to \pi$ so that the argument should give correctly the critical value of $\Lambda$.

## 4. Viscous tails

Separation can occur in a Hele-Shaw cell when (roughly) $\Lambda$ is larger than 1. Riegels' experiments show this and the analysis of §3 provides some theoretical evidence. It follows that circulation can be generated by a lifting body, with separation at a sharp trailing edge enforcing the Kutta condition. If we rewrite (10) in the form

$$p+\tfrac{1}{2}q_s^2+N\phi = \text{constant} \tag{15}$$

and consider a circuit around the surface of the airfoil it follows that $q_s$ cannot be continuous around the airfoil, since $\phi$ is not single-valued. The only plausible picture is given by figure 2. The circulation $\Gamma$ generates a shear layer attached to

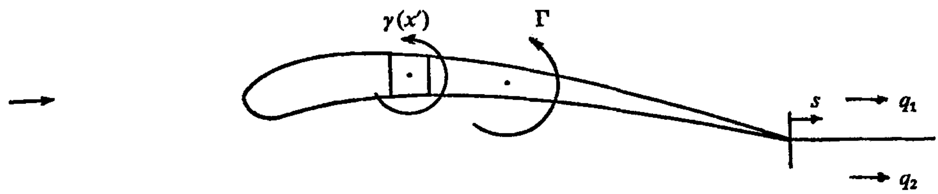


FIGURE 2. Lifting airfoil.

the trailing edge. If the airfoil is slender so that a linear analysis is appropriate this sheet has initial strength

$$\delta q = q_{s1}-q_{s2} = N\Gamma.$$

Differentiation of (15) shows that the sheet decays exponentially, i.e.

$$\delta q = N\Gamma e^{-Ns}, \tag{16}$$

where $s$ is the distance measured from the trailing edge. Note that the total circulation around the airfoil and the shear layer is zero. In addition, the decay law (16) is not the same as the linear decay law appropriate for a streamline bounding a separation bubble.

Shear layers or viscous tails of the form (16) occur in the MHD problem. They were explicitly noticed for the first time by Ludford & Fan (1969). Previous analyses (Salathe & Sirovich 1967) of the linearized lifting airfoil problem were correct, however, since the airfoil was represented as a pressure distribution rather than a vortex sheet. The pressure is continuous across the tail.†

The effects of the viscous tail on the flow field can be easily found by the usual lifting line theory. The airfoil is represented as a vortex sheet with local strength $\gamma$ per unit length (figure 2). The vertical velocity at a point on the surface of the airfoil is then given by

$$v = \frac{1}{2\pi}(P)\int_{-1}^{+1}\frac{\gamma(x')\,dx'}{x-x'}+\frac{1}{2\pi}\int_0^\infty ds\frac{N\Gamma e^{-Ns}}{s+1-x}, \tag{17}$$

where $(P)$ denotes the principal value.

† A somewhat different kind of viscous tail occurs in two-dimensional MHD flow where the applied field is perpendicular to the plane. It is well known that this flow is potential. The appropriate Bernoulli equation is

$$\tfrac{1}{2}q^2+p+\chi = \text{constant},$$

where $\chi$ is the current stream function. A tail will be generated if the body is a net source of current, but this tail has constant strength and does not decay.

The simplest problem is that of a flat plate inclined at $\alpha$ to the free stream. Equation (17) then has solution

$$\gamma(x) = -\frac{1}{\pi^2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} (P)\int_{-1}^{+1} h(t)\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} \frac{dt}{t-x}, \tag{18}$$

where

$$h(t) = 2\pi\left[\alpha + \frac{N\Gamma}{2\pi}\int_0^\infty ds\, \frac{e^{-Ns}}{s+1-t}\right].$$

The total circulation round the airfoil is

$$\Gamma = \int_{-1}^{+1} \gamma(x')\, dx',$$

whence from (18)

$$\Gamma = -\frac{1}{\pi}\int_{-1}^{+1} h(t)\left(\frac{1+t}{1-t}\right)^{\frac{1}{2}} dt$$

and on evaluating the integral

$$\Gamma = \frac{-2\pi\alpha}{1+N\int_0^\infty ds\, e^{-Ns}\left[\left(\frac{s+2}{s}\right)^{\frac{1}{2}}-1\right]}, \tag{19}$$

and as $N \to \infty$

$$\Gamma \sim \alpha/N^{\frac{1}{2}}.$$

The integral, which is positive, can be evaluated in terms of Whittaker functions. The downwash generated by the attached vortex sheet reduces the clockwise circulation around the airfoil.

## 5. Concluding remarks

It would be very interesting to observe these viscous tails in Hele-Shaw flow experimentally. To do this, it would be important to choose $\Lambda$ correctly. If $\Lambda$ is very small, equations (16) and (19) imply that the sheet has strength $O(\Lambda^{-\frac{1}{2}})$ but at the same time it decays in a distance $O(\Lambda)$. Furthermore, if $\Lambda$ is too small the Kutta condition may not be satisfied at the trailing edge and there may be no circulation at all. On the other hand, if $\Lambda$ is large, so that good circulation is generated, the tail only has strength $O(\Lambda^{-1})$. It would seem that we would need $N \sim 1$, i.e. $\Lambda \sim 4$.

Dr C. J. Wood of the Department of Engineering Science, Oxford, was kind enough to make his Hele-Shaw cell available for examination. One thing we were able to establish, in agreement with the theory, is that when $\Lambda \sim 5$, so that the flow over a circular cylinder is well separated, then there is no rotation in the cavity behind the body. This was established by injecting dye into the cavity and watching its subsequent behaviour. The cavity fluid was not completely stagnant but its speed was an order of magnitude slower than the free stream and showed no evidence of reversal.

Our search for the tail was not successful. It was confounded by the tendency for a separation cavity to form on the upper surface of the airfoil towards the trailing edge. Since watching the movements of the dye or small particles of dirt, was our only means of studying the velocity field, no conclusions could be drawn. On the credit side again though, we were able to confirm Riegels' result that separation on a circular cylinder can be suppressed when $\Lambda \sim 1$. No effort was made to measure precisely the critical value however.

The discussion should not be finished perhaps without mentioning that implicit in the picture of the viscous tail is the absence of a rear stagnation point. This is true for the MHD problem discussed by Ludford & Fan (1969) as well as the Hele-Shaw flow. Whether the 'inviscid' flow is singular or whether three-dimensional effects became locally important is not known.

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## REFERENCES

HELE-SHAW, H. S. 1897 *Trans. Roy. Inst. Nav. Arch.* **41**, 21.
HELE-SHAW, H. S. 1898*a* *Trans. Roy. Inst. Nav. Arch.* **42**, 49.
HELE-SHAW, H. S. 1898*b* *Rept. 68th Mtg. Br. Ass.* p. 136.
LUDFORD, G. S. S. & FAN, D. N. 1969 *I.I.T. J. Math. Phys. Sci.*
RIEGELS, F. 1938 *Zeit. Ang. Math. Mech.* **18**, 95.
SALATHE, P. & SIROVICH, L. 1967 *Phys. Fluids*, **10**, 1477.
STOKES, G. G. 1898 Appendix to Hele-Shaw (1898*b*).
THOMPSON, B. W. 1968 *J. Fluid. Mech.* **31**, 379.